SYDNEY GRAMMAR SCHOOL

NAME ______________________ MATHEMATICS MASTER ______________

2021 Trial HSC Examination

Form VI Mathematics Extension 2

Friday 20th August 2021

8:40am

General Instructions

- Reading time — 10 minutes
- Working time — 3 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Total Marks: 100

Section I (10 marks) Questions 1 – 10

- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.
- Write your name and master on each page.

Section II (90 marks) Questions 11 – 16

- Because of the nature of this task, greater weight than normal will be placed on working. Clear reasoning and full calculations are required.
- Record your answers on the writing paper provided.
- Start each question on a new page.
- Write your name and master on each page.

Your sheets must be ORDERED
then scanned and uploaded in a
SINGLE PDF FILE
to the Schoology page of your mathematics class

6A: RCF	6B: BDD	6C: LRP/GMC
6D: RR	6E: DNW	

Checklist

- Reference sheet
- Writing paper and multiple-choice answer sheet.

Writer: RCF

Section I

Questions in this section are multiple-choice.
Choose the single best answer for each question and record it on the provided answer sheet.

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1. The two lines with the vector equations given below intersect at the point $(-1, 3, 4)$.
   What is the exact value of the acute angle between these two lines?
   $$\underset{\sim}{r}_1 = \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{r}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$
   (A) $\cos^{-1}\left(\dfrac{1}{\sqrt{26}}\right)$
   (B) $\cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$
   (C) $\cos^{-1}\left(\dfrac{4}{9}\right)$
   (D) $\cos^{-1}\left(\dfrac{\sqrt{26}}{6}\right)$

2. Which of the following is obtained when $\dfrac{1+i}{1-i}$ is expressed in modulus-argument form?
   (A) $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$
   (B) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
   (C) $\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$
   (D) $\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

3. Consider the statement: *If you like to play board games then you are cool.*
   Which of the statements listed below is a correct version of the contrapositive?
   (A) If you like to play board games then you are not cool.
   (B) If you are cool then you like to play board games.
   (C) If you do not like to play board games then you are cool.
   (D) If you are not cool then you do not like to play board games.

4. The displacement of a particle performing simple harmonic motion is given by
   $x = 4\cos\left(\pi t - \dfrac{\pi}{4}\right) + 2$, where x is measured in metres and time t in seconds. What is the acceleration of the particle as it passes through the origin?
   (A) $4\pi^2$
   (B) $-4\pi^2$
   (C) $2\pi^2$
   (D) $-2\pi^2$

5. Which of the following gives the value of $\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^{2021}$?

   (A) $\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$

   (B) $\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$

   (C) $-\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$

   (D) $-\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$

6. Which expression is equivalent to $\displaystyle\int 2\sin^{-1} x \, dx$?

   (A) $2\sin^{-1} x - \displaystyle\int \frac{2}{\sqrt{1-x^2}}\, dx$

   (B) $\dfrac{2}{\sqrt{1-x^2}} - \displaystyle\int 2\sin^{-1} x \, dx$

   (C) $2x\sin^{-1} x - \displaystyle\int \frac{2x}{\sqrt{1-x^2}}\, dx$

   (D) $\dfrac{2x}{\sqrt{1-x^2}} - \displaystyle\int 2x\sin^{-1} x \, dx$

7. The vectors $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are such that $|\underset{\sim}{p}| = 4$, $|\underset{\sim}{q}| = 5$ and $\underset{\sim}{p} \cdot \underset{\sim}{q} = 7$. What is the value of $|\underset{\sim}{p} - \underset{\sim}{q}|$?

   (A) 2

   (B) $2\sqrt{2}$

   (C) 3

   (D) $3\sqrt{3}$

8. James makes the false claim that $n^2 - n + 1$ is a prime number, for all positive integers n. Which of the following could NOT be used as a counter-example to disprove his conjecture?

   (A) $n = 1$

   (B) $n = 3$

   (C) $n = 5$

   (D) $n = 8$

9. Consider a projectile being fired through a resistive medium. The following statements are made to describe the motion at the instant when the projectile reaches the highest point of its trajectory.

   Statement I: *The horizontal displacement of the projectile is half of the final range.*

   Statement II: *The speed of the projectile is zero.*

   Statement III: *The projectile experiences zero acceleration.*

   Which of the following options is correct?

   (A) None are true.

   (B) Only I is true.

   (C) Only II is true.

   (D) Only III is true.

10. Consider the three definite integral statements given below.

    Statement I: $\displaystyle\int_{-1}^{1} e^{-\frac{1}{2}x^2}\,dx = 0$

    Statement II: $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^3 x \sec^3 x\,dx = 0$

    Statement III: $\displaystyle\int_{0}^{\pi} \cos^3 x\,dx = 0$

    Use your knowledge of the properties of definite integrals to determine which of the following options is correct.

    (A) All three are true.

    (B) I is false but II and III are true.

    (C) I and III are false but II is true.

    (D) All three are false.

End of Section I

The paper continues in the next section

Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Record your answers on the writing paper provided.

QUESTION ELEVEN (15 marks) Start a new page. **Marks**

(a) Given two complex numbers, $z = -3 + 4i$ and $w = 1 - 3i$, find:

(i) $z - 2w$ 1

(ii) zw 1

(iii) $\dfrac{1}{z}$ 1

(iv) $\overline{z + iw}$ 1

(b) Find $\displaystyle\int \frac{4x^3}{1+x^8}\,dx$. 2

(c) Use de Moivres Theorem to simplify $\left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^5 - \left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)^5$. 2

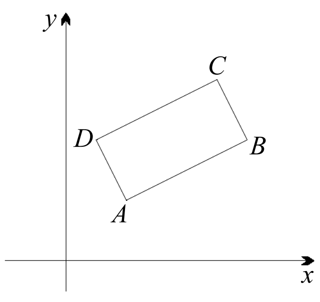
(d) The polynomial $P(z) = z^4 - 6z^3 + 14z^2 - 24z + 40$ has four complex zeroes, two of which are $2i$ and $3 - i$.

(i) Write down the other two zeroes, giving a clear justification for your answer. 1

(ii) Hence write $P(z)$ as a product of two quadratic factors with real coefficients. 1

(e) By expressing an odd integer as $2n-1$ where n is a positive integer, prove that if we subtract one from the square of an odd number the result is a multiple of eight. 2

(f)



The diagram above shows a rectangle $ABCD$ in the Argand diagram. The rectangle is twice as long as it is wide. The points A and B represent the complex numbers $2 + 2i$ and $6 + 4i$ respectively. Find the complex numbers corresponding to each of the following:

(i) $\overrightarrow{AB}$ 1

(ii) $\overrightarrow{AD}$ 1

(iii) point C 1

QUESTION TWELVE (15 marks) Start a new page. **Marks**

(a) Prove by contradiction that if p and q are positive integers, then **3**

$$\frac{p}{q} + \frac{q}{p} \geq 2.$$

(b) A particle which was initially at rest, subsequently moves in a straight line along the x-axis. Its acceleration is given by $\ddot{x} = (2v + 1)\ \mathrm{m\,s^{-2}}$, where $v\ \mathrm{m\,s^{-1}}$ is the velocity of the particle after t seconds. **2**

By expressing $\ddot{x}$ as $\dfrac{dv}{dt}$, find an expression for $v(t)$.

(c) A flight controller is monitoring the progress of three helicopters as they move along straight flight paths through Sydney airspace. Let the origin O be the control tower of Kingsford-Smith Airport. All distances are measured in kilometres with the Cartesian axes Ox, Oy and Oz oriented due east, due north and vertically upwards respectively.

The positions of the three helicopters are represented by the points $P(3t - 2, 5 - t, \frac{3}{2} + \frac{t}{4})$, $Q(t - 1, 2t - 7, 1 + \frac{t}{5})$ and $R(8 - 2t, t + 1, 3 - \frac{t}{2})$ respectively, where t represents the time in minutes after midday.

(i) Describe the location of the first helicopter relative to the airport control tower at midday. **1**

(ii) Find the direction vectors of the flight paths of each of the three helicopters. **1**

(iii) Helicopters P and Q pass through a common point A in space, but at different times. Find the coordinates of this point and the time each pilot reaches this location. **3**

(iv) Show that the third helicopter R is also on course to pass through point A and identify which of the other two helicopters it will collide with, unless the pilots are instructed to alter their courses. **1**

(d) Consider the following pair of inequations in the complex plane.

$$|z - 2 - i| \leq 1 \qquad 0 \leq \text{ Arg } z < \frac{\pi}{4}$$

(i) Sketch the intersection of the regions given by the two inequations, making sure to clearly indicate whether the boundaries and their points of intersection are included in the required region. **3**

(ii) Hence calculate the maximum value of $|z|$. **1**

QUESTION THIRTEEN (15 marks) Start a new page. **Marks**

(a) An object moves so that its displacement x metres at time t seconds is given by:

$$x = \cos 3t + 2\sin 3t.$$

(i) Show that the motion is simple harmonic by showing that it satisfies the differential equation $\ddot{x} = -n^2 x$, for some $n > 0$. **1**

(ii) Express x in the form $R\sin(3t + \alpha)$, where $R > 0$ and $0 \le \alpha < \frac{\pi}{2}$. **2**

(iii) Hence find at what time, to the nearest hundredth of a second, the object first passes through $x = 2$. **1**

(b) It is proposed, for all integers x, that:

if $x^2 - 6x + 5$ is even, then x is odd.

(i) Write down a contrapositive statement. **1**

(ii) Use contraposition to prove the original conjecture. **2**

(c) (i) Determine constants A and B such that $\dfrac{5}{(x-1)(3x+2)} = \dfrac{A}{x-1} + \dfrac{B}{3x+2}$. **1**

(ii) Hence find the particular solution of the differential equation **3**

$$v\frac{dv}{dx} = \frac{5v^2}{(x-1)(3x+2)} \qquad \text{for } \; x > 1,$$

for which $v(2) = 8$. Give your answer in the form $v = f(x)$.

(d) (i) Use proof by induction to prove de Moivres Theorem, that **3**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers $n \ge 1$.

(ii) Hence show that $\dfrac{\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right)^{11}}{\left(\cos\frac{\pi}{36} + i\sin\frac{\pi}{36}\right)^{4}}$ is purely imaginary. **1**

QUESTION FOURTEEN (15 marks) Start a new page. **Marks**

(a) Use a suitable substitution to evaluate $\displaystyle\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$. 2

(b) Consider $\dfrac{(3+4i)(1+2i)}{1+3i} = (1+i)q$, where q is a real number.

(i) Find the value of q. 1

(ii) Hence explain why $\tan^{-1}\left(\dfrac{4}{3}\right) + \tan^{-1} 2 - \tan^{-1} 3 = \dfrac{\pi}{4}$. 1

(c) The sea level in a harbour is assumed to rise and fall with simple harmonic motion. On a certain day low tide occurs at 1500 hours when the depth of the water will fall to 8 metres. The next high tide will occur at 0330 hours with a depth of 18 metres. 4

A ship wishes to enter the harbour that evening and needs a minimum depth of 12 metres of water to do so safely. Calculate, to the nearest minute, the earliest time it can enter the harbour that evening and the time by which it must leave the following morning.

(d) Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \sec^n x\,dx$.

(i) Use integration by parts to show that for $n > 2$, 3

$$I_n = \frac{1}{n-1}\left(\left(\sqrt{2}\right)^{n-2} + (n-2)I_{n-2}\right).$$

(ii) Show that $\dfrac{d}{dx}\Big(\ln\left(\sec x + \tan x\right)\Big) = \sec x$. 1

(iii) Hence show that $\displaystyle\int_0^{\frac{\pi}{4}} \sec x\,dx = \ln\left(\sqrt{2}+1\right)$ and find a simplified expression for I_5. 3

QUESTION FIFTEEN (15 marks) Start a new page. **Marks**

(a) Bernouilli's inequality famously states that, $\boxed{3}$

if $x \geq -1$ then $(1+x)^n \geq 1+nx$, for all positive integers n.

Prove this inequality using mathematical induction.

(b) (i) By applying de Moivres theorem to $(\cos\theta + i\sin\theta)^3$, show that $\boxed{1}$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$$

(ii) Hence, or otherwise, find an expression for $\tan 3\theta$ in terms of $\tan\theta$. $\boxed{2}$

(iii) Show that $\tan\dfrac{\pi}{12}$ is a root of the cubic equation $\boxed{2}$

$$x^3 - 3x^2 - 3x + 1 = 0$$

and find two other values of θ for which $\tan\theta$ is a root of this equation, where $0 < \theta < \pi$.

(iv) Hence show that $\boxed{1}$

$$\tan\frac{\pi}{12} + \tan\frac{5\pi}{12} = 4.$$

(c) A particle of mass m is projected vertically upwards with initial speed u metres per second and experiences a resistance proportional to its speed v. That is, a resistive force of mkv Newtons acts in an opposite direction to its motion, where k is a positive constant. After T seconds the particle attains its maximum height H metres.

Let the acceleration due to gravity be g m/s^2. Define the origin as the point of projection and upwards as the positive direction for the motion. The acceleration of the particle can thus be given by the differential equation

$$\ddot{y} = -(g + kv),$$

where y is the height of the particle t seconds after the launch.

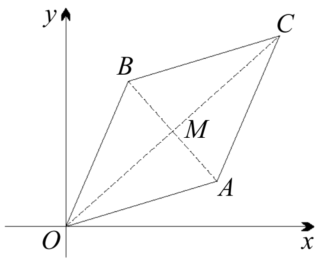
(i) Prove that $T = \dfrac{1}{k}\ln\left(\dfrac{g+ku}{g}\right)$. $\boxed{2}$

(ii) Show that $H = \dfrac{u - gT}{k}$. $\boxed{3}$

(iii) Before considering the downward journey we will define a new origin as the point reached when the particle is at its maximum height. Also define downwards as the new positive direction for the motion. Using this revised coordinate system, write down the new differential equation for the acceleration of the particle during its downward journey. $\boxed{1}$

QUESTION SIXTEEN (15 marks) Start a new page. **Marks**

(a)



Let $z_1 = \operatorname{cis}\alpha$ and $z_2 = \operatorname{cis}\beta$ where $0 < \alpha < \beta < \frac{\pi}{2}$. The complex numbers z_1, z_2 and $(z_1 + z_2)$ are represented by the points A, B and C respectively in the Argand diagram above. Let $(z_1 + z_2) = r\operatorname{cis}\theta$ and let OC and AB intersect at M.

(i) State what type of quadrilateral $OACB$ is, giving a clear geometric justification for your answer. **1**

(ii) Use the properties of the quadrilateral to determine θ in terms of α and β. **1**

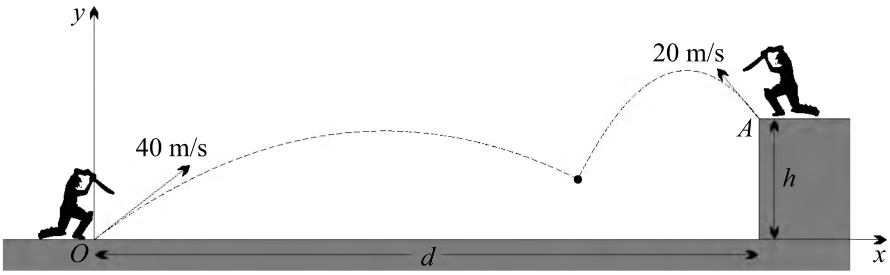
(iii) By considering triangle OAC, express r in terms of α and β. **2**

(iv) Hence show that **2**

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\beta-\alpha}{2}\right)\cos\left(\frac{\beta+\alpha}{2}\right).$$

Question 16 continues over the page

(b)



Two cricketers are trying to film a 'trick' by simultaneously each hitting a cricket ball and making the balls collide in mid-air. The first cricketer is at point O, at ground level, while the second is at the top of a nearby building at point A. The point A is a horizontal distance d metres away from point O and at a height h metres above the horizontal ground, as shown in the diagram above. 4

The ball hit from O has an initial speed of 40 m/s and is given an angle of projection of $\tan^{-1}\left(\frac{3}{4}\right)$. Define the origin at O with right and upwards as the positive directions for x and y respectively, as shown in the diagram above. It then follows that the six equations describing the motion of the ball hit by the first cricketer, t seconds after release are:

$$\ddot{x} = 0 \qquad \dot{x} = 32 \qquad x = 32t$$

$$\ddot{y} = -g \qquad \dot{y} = 24 - gt \qquad y = 24t - \frac{1}{2}gt^2$$

The other cricketer hits the second ball at the same instant, from point A towards O, with an initial speed of 20 m/s and an angle of projection of $\tan^{-1}\left(\frac{4}{3}\right)$. Assume that the motion of the two balls takes place in the same vertical plane, they behave as particles, experience no air resistance and that the cricketers are successful in making the two balls collide during their flights.

Find the comparable set of six equations for the motion of the second cricket ball and hence find the ratio of the two lengths d and h.

(c) Consider the improper integral $I = \displaystyle\int_1^{\infty} \frac{1}{1+x^3}\,dx$.

(i) Use the substitution $u = \dfrac{1}{x}$ to show that $I = \displaystyle\int_0^{1} \frac{u}{1+u^3}\,du$. 2

(ii) Hence or otherwise evaluate $\displaystyle\int_0^{\infty} \frac{1}{1+x^3}\,dx$. 3

END OF PAPER

2021 MATHS EXT 2 TRIAL SOLUTIONS

Multiple Choice

1) C
2) A
3) D
4) C
5) D
6) C
7) D
8) B
9) A
10) B

① Use direction vectors $\underset{\sim}{b}_1 = \begin{bmatrix}-1\\-2\\2\end{bmatrix}$ $\underset{\sim}{b}_2 = \begin{bmatrix}-2\\1\\2\end{bmatrix}$

$\underset{\sim}{b}_1 \cdot \underset{\sim}{b}_2 = 2 - 2 + 4$

$= 4$

$\sqrt{9}\sqrt{9}\cos\theta = 4$

$\cos\theta = 4/9$

$\theta = \cos^{-1}(4/9)$ (C)

② $\frac{1+i}{1-i} \times \frac{(1+i)}{(1+i)} = \frac{1+2i+i^2}{1-i^2}$

$= \frac{2i}{2}$

$= i$

$= cis\left(\frac{\pi}{2}\right)$ (A)

③ If $p \Rightarrow q$

Contrapositive $\neg q \Rightarrow \neg q$

hence (D)

④ $x = 4\cos\left(\pi t - \frac{\pi}{4}\right) + 2$

$\dot{x} = -4\pi \sin\left(\pi t - \frac{\pi}{4}\right)$

$\ddot{x} = -4\pi^2 \cos\left(\pi t - \frac{\pi}{4}\right)$ (C)

At origin $x = 0$ hence

$\cos\left(\pi t - \frac{\pi}{4}\right) = \left(-\frac{1}{2}\right)$

hence $\ddot{x} = -4\pi^2\left(-\frac{1}{2}\right)$

$= 2\pi^2$

Alternatively Use Diff Eqn When $x = 0$

$x = 4\cos\left(\pi t - \frac{\pi}{4}\right) + 2$

$\ddot{x} = -\pi^2(x-2)$

$\ddot{x} = +2\pi^2$

⑤ $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = 1 cis \frac{\pi}{4}$

$\left(cis\frac{\pi}{4}\right)^{2021} = cis\left(\frac{2021\pi}{4}\right)$ By De Moivres

$= cis\left(\frac{5\pi}{4}\right)$ $\left(\text{since } \frac{2021\pi}{4} = \frac{2016\pi}{4} + \frac{5\pi}{4} = 252 \times (2\pi) + \frac{5\pi}{4}\right)$

$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

(D)

⑥ By parts $u = \sin^{-1}x$ $v' = 2$

$u' = \frac{1}{\sqrt{1-x^2}}$ $v = 2x$ (C)

$\int 2\sin^{-1}x\,dx = 2x\sin^{-1}x - \int \frac{2x}{\sqrt{1-x^2}}\,dx$

⑦ $|\underset{\sim}{p} - \underset{\sim}{q}|^2 = (\underset{\sim}{p} - \underset{\sim}{q}) \cdot (\underset{\sim}{p} - \underset{\sim}{q})$

$= \underset{\sim}{p}\cdot\underset{\sim}{p} - \underset{\sim}{q}\cdot\underset{\sim}{p} - \underset{\sim}{p}\cdot\underset{\sim}{q} + \underset{\sim}{q}\cdot\underset{\sim}{q}$

$= |\underset{\sim}{p}|^2 + |\underset{\sim}{q}|^2 - 2\underset{\sim}{p}\cdot\underset{\sim}{q}$

$= 4^2 + 5^2 - 2 \times 7$

$= 16 + 25 - 14$

$= 27$

$|\underset{\sim}{p} - \underset{\sim}{q}| = \sqrt{27}$

$= 3\sqrt{3}$ (D)

⑧ If $n = 1$ $n^2 - n + 1 = 1$ NOT PRIME

$n = 3$ $n^2 - n + 1 = 7$ PRIME

$n = 5$ $n^2 - n + 1 = 21$ NOT PRIME (B)

$n = 8$ $n^2 - n + 1 = 57 = 3 \times 19$ NOT PRIME

$n = 1, 5, 8$ could all be used as counter examples

hence $n = 3$ could NOT

⑨ Trajectory is no longer parabolic, nor asymmetric I - False

Projectile still has horizontal velocity only vertical component is zero II - False (A)

Still experiences non-zero net force hence accelerating III - False

⑩ $\forall x$, $e^{-x^2/2} > 0$ ∴ Integral positive NOT zero

I is FALSE

$\tan x$ is odd fn, $\sec x$ even function

hence $\tan^3 x \sec^3 x$ is odd. Interval is symmetric

hence integral is zero

II is TRUE

$\cos x$ has point symmetry at $x = \frac{\pi}{2}$

hence $\cos^3 x$ has point symmetry at $x = \frac{\pi}{2}$

hence integral is zero.

III is TRUE

(B)

QUESTION 11

a) (i) $(-3+4i)-2(1-3i) = \underline{-5+10i}$ ✓

(ii) $(-3+4i)(1-3i) = -3+4i+9i-12i^2$
$= \underline{9+13i}$ ✓

(iii) $\dfrac{1}{-3+4i} \dfrac{\times(-3-4i)}{\times(-3-4i)} = \underline{\dfrac{1}{25}(-3-4i)}$ ✓

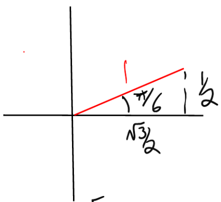
(iv) $\overline{z+iw} = \overline{-3+4i+i(1-3i)}$
$= \overline{0+5i}$
$= \underline{-5i}$ ✓

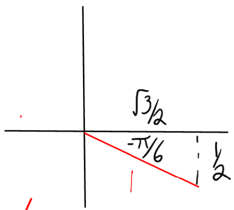
b) Recognise $u = x^4$
$\dfrac{du}{dx} = 4x^3$
hence reverse chain rule
$= \underline{\tan^{-1} x^4 + C}$ ✓✓

1 mark $\tan^{-1}$
2 mark x^4
(No penalty for omission of $+C$)

Alternatively: Full substitution
$u = x^4$
$du = 4x^3\,dx$
$\frac{1}{4}du = x^3\,dx$ ✓
$I = \frac{1}{4}\int \frac{1}{1+u^2}\,du$
$= \frac{1}{4}\tan^{-1} u + C$
$= \underline{\frac{1}{4}\tan^{-1} x^4 + C}$ ✓

c)





$\left(\text{cis}\frac{\pi}{6}\right)^5 - \left(\text{cis}\left(-\frac{\pi}{6}\right)\right)^5$ ✓
$= \text{cis}\left(\frac{5\pi}{6}\right) - \text{cis}\left(-\frac{5\pi}{6}\right)$
$= \left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} - \left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)\right)$
$= 2i\sin\frac{5\pi}{6}$
$= \underline{i}$ ✓ $\left(\text{since } \sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}\right)$

d) (i) Other two roots are $\underline{-2i \text{ and } 3+i}$
Since complex roots appear in conjugate pairs when coefficients of polynomial are REAL ✓
Reason and Answer

ii) $P(z) = (z+2i)(z-2i)(z-(3-i))(z-(3+i))$
$= \underline{(z^2+4)(z^2-6z+10)}$ ✓

$\left(\begin{aligned} \text{Check}: & \; z^4 - 6z^3 + 10z^2 \\ & + 4z^2 - 24z + 40 \\ = & \; z^4 - 6z^3 + 14z^2 - 24z + 40 \end{aligned}\right)$

e) Let odd integer be $2n-1$, $n \in \mathbb{N}$
Conjecture $(2n-1)^2 - 1$ is a multiple of 8
$(2n-1)^2 - 1 = 4n^2 - 4n + 1 - 1$
$= 4n^2 - 4n$
$= 4n(n-1)$ ✓
since $n \in \mathbb{Z}^+$, $n(n-1)$ is product of two consecutive integers one of which must be even and other odd hence product is EVEN ie $n(n+1) = 2m$ $m \in \mathbb{N}$
$(2n+1)^2 - 1 = 4 \times 2m$
$= 8m$ ✓
hence a multiple of 8.

(Alternatively: if argued with words BUT without $(2n-1)$ award ½ for complete proof)

f) (i) $\vec{OA}$ is $2+2i$ $\quad \vec{AB} = \vec{OB} - \vec{OA}$ ✓
$\vec{OB}$ is $6+4i$ $\quad \vec{AB}$ is $\underline{4+2i}$

(ii) $\vec{AD} = \vec{BC}$ (Equal Parallel sides of rectangle)
$\vec{AD}$ is $\vec{BC}$ rotated $90°$ anticlockwise and halved.
$\vec{AD}$ is $\frac{i}{2} \times (4+2i) = \underline{-1+2i}$ ✓

(iii) $\vec{OC} = \vec{OB} + \vec{BC}$ $\quad C$ is $6+4i+(-1+2i)$ ✓
$= \vec{OB} + \vec{AD}$ $\quad = \underline{5+6i}$

Alternatively: $\vec{AC} = \vec{AB} + \vec{BC}$ $\quad \vec{AC}$ is $3+4i$
$= \vec{AB} + \vec{AD}$
$\vec{OC} = \vec{OA} + \vec{AC}$ $\quad$ pt C is $2+2i+3+4i$
$= \underline{5+6i}$

QUESTION 12

a) By way of contradiction, assume $\frac{p}{q}+\frac{q}{p}<2$ ✓

$(\times pq)$ since $p,q \in \mathbb{Z}^+$

$p^2+q^2<2pq$

$p^2-2pq+q^2<0$ ✓

$(p-q)^2<0$

This is a contradiction since a square is by definition non-negative. ✓

Hence $\frac{p}{q}+\frac{q}{p}\geq 2$ using proof by contradiction

b) $\frac{dv}{dt}=2v+1$

$\frac{dt}{dv}=\frac{1}{2v+1}$

$\left(\int dv\right)$ $t=\frac{1}{2}\ln|2v+1|+C$ ✓

$t=0 \quad v=0 \quad \Rightarrow C=0$

$t=\frac{1}{2}\ln|2v+1|$

$e^{2t}=2v+1$

$v=\frac{1}{2}(e^{2t}-1)$ ✓

c) (i) At midday $t=0$

$P(-2,5,\frac{3}{2})$

First helicopter is 2km west, 5km north and at an altitude of $1\frac{1}{2}$km. ✓

(ii) Direction Vector of first helicopter P is $\begin{bmatrix}3\\-1\\1/4\end{bmatrix}$

Similarly for Q, direction vector is $\begin{bmatrix}1\\2\\1/5\end{bmatrix}$

for R direction vector is $\begin{bmatrix}-2\\1\\-1/2\end{bmatrix}$ ✓ (Allow 1 numerical/sign error but 2 correct vectors for mark)

(iii) If helicopter P & Q pass through same point but at different times replace parameter t with λ and μ respectively

Find where paths of helicopters P & Q cross

$$\begin{bmatrix}3\lambda-2\\5-\lambda\\3/2+\lambda/4\end{bmatrix}=\begin{bmatrix}\mu-1\\2\mu-7\\1+\mu/5\end{bmatrix}$$

$3\lambda-2=\mu-1$ ①

$5-\lambda=2\mu-7$ ② ✓

$2\times$① $-$ ② $7\lambda-9=5$

$\lambda=2$

sub into ① $4=\mu-1$

$\mu=5$ ✓

check in ② LHS $5-\lambda=3$

RHS $2\mu-7=3$

Should confirm consistent with z co-ordinate as well to ensure lines not SKEW

$\frac{3}{2}+\frac{\lambda}{4}=1+\frac{\mu}{5}$ ③

LHS $\frac{3}{2}+\frac{2}{4}=2$

RHS $1+\frac{5}{5}=2$

hence paths of P & Q cross at A(4,3,2) with P passing through at 12:02pm and Q at 12:05pm ✓ (Must confirm consistency with 3rd co-ord not used to solve for λ & μ to gain full marks)

Confirm helicopter R passes through A

$$\begin{bmatrix}8-2t\\t+1\\3-t/2\end{bmatrix}=\begin{bmatrix}4\\3\\2\end{bmatrix}$$

$8-2t=4$ ① $\quad 2t=4 \quad t=2$

$t+1=3$ ② $\quad t=2$

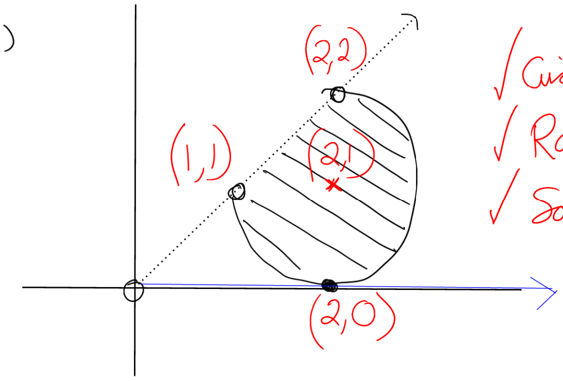
$3-\frac{t}{2}=2$ ③ $\quad 1=t/2 \quad 2=t$

hence helicopter R passes through A at 12:02pm

and will crash with P unless one changes course

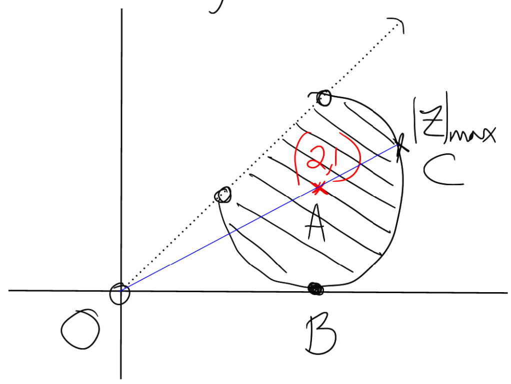
✓ Must show t=2 for all 3 eqns
Else substitute t=2 into R and confirm R(4,3,2) for the mark since "SHOW that"

d) (i)



✓ Circle Centre (2,1) r = 1
✓ Rays x axis & main diagonal
✓ Solid circle boundary, Dotted ray, Hollow dots at (1,1) (2,2).

(ii) Max value of $|z|$ occurs for ray from origin passing through centre of circle ie pt C



$OA = \sqrt{1^2+2^2}$
$= \sqrt{5}$
$OC = 1+\sqrt{5}$
$|z|_{max} = 1+\sqrt{5}$ ✓

QUESTION 13

a) (i) $x = \cos 3t + 2\sin 3t$
$\dot{x} = -3\sin 3t + 6\cos 3t$
$\ddot{x} = -9\cos 3t - 18\sin 3t$
$= -9(\cos 3t + 2\sin 3t)$
$= -9x$ ✓ SHOW
hence SHM with $n=3$

(ii) Let $R\sin(3t+\alpha) = \cos 3t + 2\sin 3t$
$R\sin 3t\cos\alpha + R\cos 3t\sin\alpha = \cos 3t + 2\sin 3t$
Equate coeffs of $\sin 3t$ $R\cos\alpha = 2$ ①
$\cos 3t$ $R\sin\alpha = 1$ ② ✓
$①^2 + ②^2$ $R^2(\cos^2\alpha + \sin^2\alpha) = 4+1$
$R = \sqrt{5}$ $R>0$
hence $\sin\alpha = \frac{1}{\sqrt{5}}$ $\cos\alpha = \frac{2}{\sqrt{5}}$ $\alpha = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$ or $\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$ or $\tan^{-1}\left(\frac{1}{2}\right)$
$x = \sqrt{5}\sin\left(3t + \sin^{-1}\frac{1}{\sqrt{5}}\right)$ ✓

(iii) $x = 2 \Rightarrow \sin(3t+\alpha) = \frac{2}{\sqrt{5}}$ $t>0$
$3t+\alpha > \sin^{-1}\frac{1}{\sqrt{5}}$
$\doteq 0.46\ldots$
$3t+\alpha = 1.107\ldots$
$3t = 0.6435\ldots$
$t = 0.2145\ldots$
$\doteq 0.21$ seconds ✓

b) (i) If $p \Rightarrow q$, Contrapositive is $\neg q \Rightarrow \neg p$
If x is NOT odd then x^2-6x+5 is NOT even ✓
OR If x is EVEN then x^2-6x+5 is ODD.

(ii) Let $x = 2n, n \in \mathbb{Z}$
$x^2-6x+5 = (2n)^2 - 6(2n) + 5$
$= 4n^2 - 12n + 5$
$= 4n(n-3) + 5$ ✓
$= 2q + 5$ where $q \in \mathbb{Z}$
hence x^2-6x+5 is ODD ✓
hence original conjecture proven by contraposition

c) (i) $\frac{5}{(x-1)(3x+2)} = \frac{A}{x-1} + \frac{B}{3x+2}$
$A = \frac{5}{3\times1+2}$ (By Cover-Up Rule)
$= 1$
$B = \frac{5}{\left(-\frac{2}{3}-1\right)}$
$= (-3)$
$= \frac{1}{x-1} - \frac{3}{3x+2}$ ✓

(ii) $v\frac{dv}{dx} = \frac{5v^2}{(x-1)(3x+2)}$ Constant solution $v=0$
$\int \frac{1}{v}\,dv = \int \frac{5}{(x-1)(3x+2)}\,dx$ ✓
$\ln|v| = \int \frac{1}{x-1} - \frac{3}{3x+2}\,dx$
$\ln|v| = \ln(x-1) - \ln(3x+2) + C$ $(x>1)$
$\ln\left|\frac{v(3x+2)}{x-1}\right| = C$ $(C \in \mathbb{R})$
$\frac{v(3x+2)}{x-1} = A$ $(A \in \mathbb{R})$
$v = \frac{A(x-1)}{3x+2}$ ✓

Do not penalise incorrect treatment of absolute values and constant

Initial conditions

$v(2)=8$ $\qquad 8=\frac{A(2-1)}{3\times2+2}$

$8=\frac{A}{8}$

$A=64$

$V=\frac{64(x-1)}{3x+2}$ ✓

d) (i) Show conjecture true for $n=1$

$LHS = (\cos\theta+i\sin\theta)^1 = \cos\theta+i\sin\theta$

$RHS = \cos\theta+i\sin\theta$

$LHS=RHS$ hence true for $n=1$

Assume true for $n=k$, $k\in\mathbb{Z}^+$ that is,

$(\cos\theta+i\sin\theta)^k = \cos k\theta+i\sin k\theta$

RTP for $n=k+1$

$(\cos\theta+i\sin\theta)^{k+1} = \cos(k+1)\theta+i\sin(k+1)\theta$ $\circledast$ ✓

$LHS = (\cos\theta+i\sin\theta)^k(\cos\theta+i\sin\theta)$

$= (\cos k\theta+i\sin k\theta)(\cos\theta+i\sin\theta)$ (by assumption $\circledast$) ✓

$= \cos k\theta\cos\theta-\sin k\theta\sin\theta+i(\sin k\theta\cos\theta+\cos k\theta\sin\theta)$

$= \cos(k+1)\theta+i\sin(k+1)\theta$ (by compound angle theorems) ✓

$= RHS$

hence if true for $n=k$ then true for $n=k+1$

By principle of mathematical induction conjecture true for all positive integer n.

(ii) $\frac{\left(\operatorname{cis}\frac{\pi}{18}\right)^{11}}{\left(\operatorname{cis}\frac{\pi}{36}\right)^4} = \frac{\operatorname{cis}\left(\frac{11\pi}{18}\right)}{\operatorname{cis}\left(\frac{\pi}{9}\right)}$

$= \operatorname{cis}\left(\frac{11\pi}{18}-\frac{2\pi}{18}\right)$ ✓ SHOW

$= \operatorname{cis}\frac{\pi}{2}$

$= i$

$\therefore$ purely imaginary

QUESTION 14

a) Let $u = \sqrt{x}$

$du = \frac{1}{2}x^{-\frac{1}{2}}dx$

$2du = \frac{dx}{\sqrt{x}}$

x	1	4
u	1	2

$$\int_1^2 2e^u du = \left[2e^u\right]_1^2$$
$$= 2(e^2 - e)$$
$$= 2e(e-1)$$

b) (i) $q = \dfrac{(3+4i)(1+2i)}{(1+3i)(1+i)}$

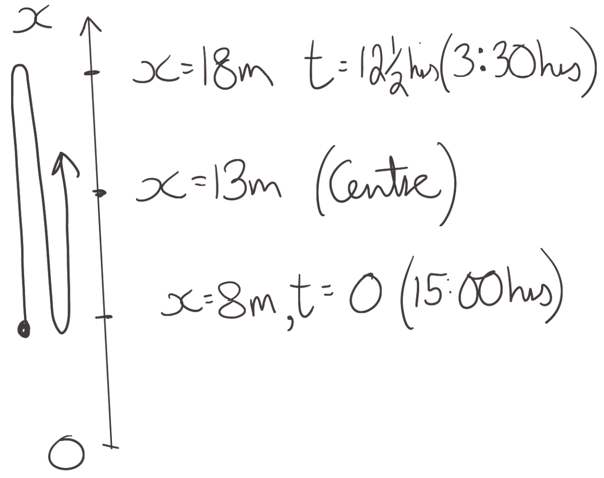
$$= \frac{3-8+4i+6i}{1-3+3i+i}$$
$$= \frac{10i-5}{4i-2}$$
$$= \frac{5\cancel{(-1+2i)}}{2\cancel{(-1+2i)}}$$
$$q = 5/2$$

(ii) From (i) $\text{Arg}\left[\dfrac{(3+4i)(1+2i)}{(1+3i)}\right] = \text{Arg}\left[\frac{5}{2}(1+i)\right]$

$$\text{Arg}(3+4i) + \text{Arg}(1+2i) - \text{Arg}(1+3i) = \frac{\pi}{4}$$
$$\tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{2}{1}\right) - \tan^{-1}\left(\frac{3}{1}\right) = \frac{\pi}{4}$$

SHOW

c)



Define x metres as depth

Let $t = 0$ hrs be 15:00

Hence amplitude of motion is 5m

Time Period is 25 hrs

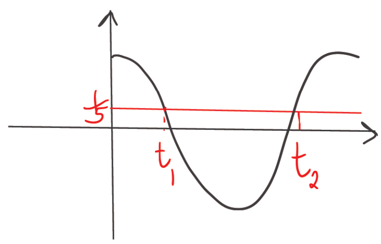
$\frac{2\pi}{n} = 25 \Rightarrow n = \frac{2\pi}{25}$

Model depth of water using $x = 13 - 5\cos\frac{2\pi t}{25}$

Minimum depth of 12m $\Rightarrow$ $5\cos\frac{2\pi t}{25} \leq 1$

$x \geq 12$

$$\cos\frac{2\pi t}{25} \leq \frac{1}{5}$$



requires $t_1 \leq t \leq t_2$

at 12m, $\frac{2\pi t}{25} = \cos^{-1}\left(\frac{1}{5}\right)$

$\doteq 1.369\ldots,\ 2\pi - 1.369$

hence

$$1.369\ldots \leq \frac{2\pi t}{25} \leq 2\pi - 1.369\ldots$$

$$5.448\ldots \leq t \leq 25 - 5.448\ldots$$

$$5.448\ldots \leq t \leq 19.551\ldots$$

5hrs 27mins $\leq t \leq$ 19hrs 33mins

Ship can safely enter harbour at 20:27 and must leave by 10:33 the following morning ✓

d) (i) $\displaystyle\int_0^{\pi/4} \sec^n x\,dx = \int_0^{\pi/4} \sec^2 x \sec^{n-2} x\,dx$

Let $u = \sec^{n-2} x$

$u' = (n-2)\sec^{n-3} x \times \sec x \tan x$

$= (n-2)\tan x \sec^{n-2} x$

$v' = \sec^2 x$

$v = \tan x$

$$I_n = \left[\sec^{n-2} x \tan x\right]_0^{\pi/4} - \int_0^{\pi/4} (n-2)\tan^2 x \sec^{n-2} x\,dx$$

✓ CORRECT INT BY PARTS

$$= \left[\left(\frac{1}{\cos\frac{\pi}{4}}\right)^{n-2} \times \tan\frac{\pi}{4} - 0\right] - (n-2)\int_0^{\pi/4} (\sec^2 x - 1)\sec^{n-2} x\,dx$$

$$I_n = (\sqrt{2})^{n-2} - (n-2)\left[I_n - I_{n-2}\right]$$

✓ RECURRENCE

$$I_n + (n-2)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2}$$

$$(n-1)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2}$$

✓ SHOW

$$I_n = \frac{1}{n-1}\left[(\sqrt{2})^{n-2} + (n-2)I_{n-2}\right]$$ as reqd.

(ii) $$\frac{d}{dx}\left[\ln(\sec x + \tan x)\right] = \frac{1}{\sec x + \tan x} \times (\sec x \tan x + \sec^2 x)$$ (By Chain Rule)

$$= \frac{\sec x\,\cancel{(\tan x + \sec x)}}{\cancel{\sec x + \tan x}}$$

✓ SHOW

$$= \sec x$$

(iii) $$\int_0^{\pi/4} \sec x\,dx = \left[\ln(\sec x + \tan x)\right]_0^{\pi/4}$$

$$= \ln\left(\frac{1}{\cos\frac{\pi}{4}} + \tan\frac{\pi}{4}\right) - \ln(1+0)$$ ✓ SHOW

$$= \ln(\sqrt{2}+1)$$ as required

From (ii) $I_1 = \ln(\sqrt{2}+1)$

$$I_3 = \frac{1}{2}\left[\sqrt{2} + I_1\right]$$

$$I_5 = \frac{1}{4}\left[\sqrt{2}^3 + 3I_3\right]$$

$$= \frac{1}{4}\left[2\sqrt{2} + \frac{3}{2}(\sqrt{2} + I_1)\right]$$ ✓

$$= \frac{1}{4}\left[2\sqrt{2} + \frac{3\sqrt{2}}{2} + \frac{3}{2}\ln(\sqrt{2}+1)\right]$$

$$= \frac{1}{8}\left[7\sqrt{2} + 3\ln(\sqrt{2}+1)\right]$$ ✓

QUESTION 15

a) Prove true for $n=1$

$LHS = (1+x)^1 = 1+x$ $\quad$ $RHS = 1+1\times x = 1+x$

$\therefore LHS = RHS$

$LHS \geq RHS$

$\therefore$ true for $n=1$

Assume true for $n=k$ $\quad k \in \mathbb{Z}^+$

$(1+x)^k \geq 1+kx$ ⊛

Prove true for $n=k+1$, that is

$(1+x)^{k+1} \geq 1+(k+1)x$

✓ CORRECT SETUP

$LHS = (1+x)^{k+1}$

$= (1+x)(1+x)^k$

$\geq (1+x)(1+kx)$ using induction hypothesis ⊛

✓ CORRECT USE OF HYPOTHESIS

$= 1+x+kx+kx^2$

$= 1+(k+1)x + kx^2$

$\geq 1+(k+1)x$ $\quad$ since $x^2 \geq 0$ & $k \in \mathbb{Z}^+$ ✓

(Note: The induction step is NOT TRUE for $x<-1$ and actually needs careful treatment for $-1 \leq x < 0$, considering signs but for marking purposes ignore this)

hence $LHS \geq RHS$

hence if conjecture true for $n=k$ also true for $n=k+1$

By principle of mathematical induction true for all positive integers n

(i) $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta \, i\sin\theta + 3\cos\theta \, i^2\sin^2\theta + i^3\sin^3\theta$

$\cos 3\theta + i\sin 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$ / SHOW

Equating real parts $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

(ii) Equating imag parts $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ ✓

hence $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta}$

$= \dfrac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$ $\quad \begin{matrix}(\div \cos^3\theta) \\ (\div \cos^3\theta)\end{matrix}$

$= \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$ ✓

(iii) $\tan 3\theta = 1$

$3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12} = \frac{3\pi}{4}$

Let $x = \tan\frac{\pi}{12}$

$$1 = \frac{3x - x^3}{1 - 3x^2}$$

$$1 - 3x^2 = 3x - x^3$$

$$x^3 - 3x^2 - 3x + 1 = 0 \quad \text{as reqd}$$

✓ SHOW

Roots of cubic eqn are $x = \tan\theta$ $\quad \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$ ✓

(iv) $\sum \alpha = -\frac{b}{a}$ hence $\tan\frac{\pi}{12} + \tan\frac{5\pi}{12} + \tan\frac{3\pi}{4} = -\frac{(-3)}{1}$ ✓SHOW

$$\tan\frac{\pi}{12} + \tan\frac{5\pi}{12} - 1 = 3$$

$$\tan\frac{\pi}{12} + \tan\frac{5\pi}{12} = 4 \quad \text{(as required)}$$

c) (i)

$$\frac{dv}{dt} = -(g + kv)$$

$$\int_u^0 \frac{dv}{g + kv} = -\int_0^T dt \quad ✓$$

$$\left[\frac{1}{k}\ln|g + kv|\right]_u^0 = [-t]_0^T$$

$$-T = \frac{1}{k}\ln g - \frac{1}{k}\ln(g + ku) \quad ✓\text{SHOW}$$

$$T = \frac{1}{k}\left[\ln(g + ku) - \ln g\right]$$

$$T = \frac{1}{k}\ln\left(\frac{g + ku}{g}\right) \quad \text{as required}$$

(ii)

$$v\frac{dv}{dy} = -(g + kv)$$

$$\int_u^0 \frac{v\,dv}{g + kv} = -\int_0^H dy \quad ✓$$

$$[-y]_0^H = \frac{1}{k}\int_u^0 \frac{kv + g - g}{g + kv}\,dv$$

$$-H = \frac{1}{k}\int_u^0 1 - \frac{g}{g + kv}\,dv \quad ✓$$

$$H = -\frac{1}{k}\left[v - \frac{g}{k}\ln(g + kv)\right]_u^0$$

$$H = -\frac{1}{k}\left[\left(-\frac{g}{k}\ln g\right) - \left(u - \frac{g}{k}\ln(g + ku)\right)\right]$$

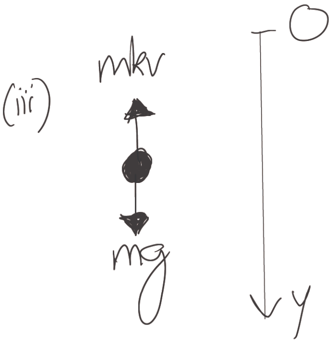
$$H = -\frac{1}{k}\left[\frac{g}{k}\ln\left(\frac{g + ku}{g}\right) - u\right] \quad ✓\text{SHOW}$$

$$= -\frac{1}{k}\left[gT - u\right]$$

$$H = \frac{1}{k}(u - gT) \quad \text{as required}$$

NB: Lots of boys worked from (i) $v(t)$ which was much more involved

(iii)

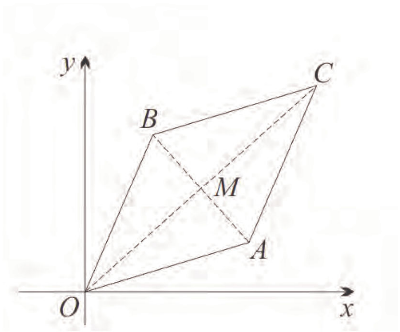


Eqn of motion ↓ $\quad mg - mkv = m\ddot{y}$

$$\ddot{y} = g - kv \quad ✓$$

QUESTION 16

a)



(i) OACB is a RHOMBUS since a parallelogram with equal adjacent sides OA = OB = 1 since z_1 & z_2 have unit moduli ✓

(ii) Diagonals of a rhombus bisect angles at vertices hence $\theta = \alpha + \frac{1}{2}(\beta - \alpha)$ ✓

$= \frac{1}{2}(\alpha + \beta)$

(iii) Cosine rule in $\triangle AOC$

$OC^2 = OA^2 + AC^2 - 2 \times OA \times AC \times \cos\angle OAC$

$= 1^2 + 1^2 - 2 \times 1 \times 1 \times [-\cos(\beta - \alpha)]$ ✓

(since $AC = OB = 1$

$\angle OAC = 180° - \angle BOA$ (Co-interior $\angle$, $OB \parallel AC$)

$= 180° - (\beta - \alpha)$

$\cos\angle OAB = -\cos(\beta - \alpha)$

(since supplementary angles opposite cosines))

$OC^2 = 2 + 2\cos(\beta - \alpha)$

$OC^2 = 2(1 + \cos(\beta - \alpha))$ ✓

$= 2 \times 2\cos^2\left(\frac{\beta - \alpha}{2}\right)$ (Using Double Angle $\cos 2\theta = 2\cos^2\theta - 1$)

$OC = 2\cos\left(\frac{\beta - \alpha}{2}\right)$

$r = 2\cos\left(\frac{\beta - \alpha}{2}\right)$

Alternatives:
1) Use cosine rule $AC^2 =$
2) Drop altitude from A to use right angled $\triangle OAM$

Use discretion here since some boys will give (iii) $r = \sqrt{2(1 + \cos(\beta - \alpha))}$ and not use double angle formulae until (iv)

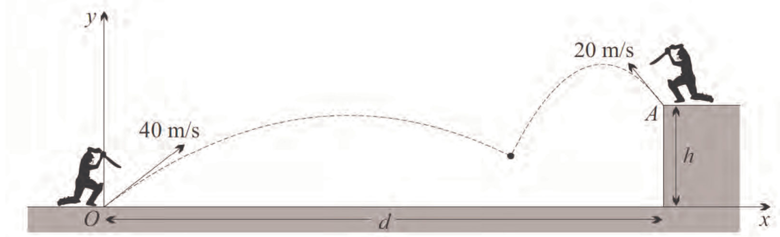
(iv) since $z_1 + z_2 = \text{cis}\,\alpha + \text{cis}\,\beta = r\,\text{cis}\,\theta$ ✓

$\cos\alpha + \cos\beta + i(\sin\alpha + \sin\beta) = 2\cos\left(\frac{\beta - \alpha}{2}\right)[\cos\theta + i\sin\theta]$ ✓SHOW

Equating real parts

$\cos\alpha + \cos\beta = 2\cos\left(\frac{\beta - \alpha}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right)$

b)



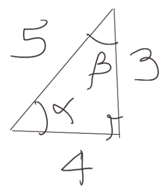
$\ddot{x} = 0 \qquad \dot{x} = 32 \qquad x = 32t$

$\ddot{y} = -g \qquad \dot{y} = 24 - gt \qquad y = 24t - \frac{1}{2}gt^2$

Taking origin at O and t = 0 at time of projection

Define angles of projection α & β at O and A respectively

$\alpha = \tan^{-1}\left(\frac{3}{4}\right) \qquad \beta = \tan^{-1}\left(\frac{4}{3}\right)$



For ball from A

$\ddot{x}_A = 0$

$\dot{x}_A = -20\cos\beta$
$= -20 \times \frac{3}{5}$
$= -12$

$x_A = d - 12t$

$\ddot{y}_A = -g$

$\dot{y}_A = 20\sin\beta - gt$
$= 20 \times \frac{4}{5} - gt$
$= 16 - gt$

$y_A = 16t - \frac{1}{2}gt^2 + h$ ✓

Balls collide at time when x displacements equal

$$32t = d - 12t$$
$$44t = d$$
$$t = \tfrac{d}{44} \quad (1)$$

✓

At this time y displacements also equal

$$24t - \tfrac{1}{2}gt^2 = 16t - \tfrac{1}{2}gt^2 + h$$
$$8t = h$$
$$t = \tfrac{h}{8} \quad (2)$$

✓

hence

$$\frac{d}{44} = \frac{h}{8}$$
$$\frac{d}{h} = \frac{44}{8}$$
$$\frac{d}{h} = \frac{11}{2}$$

ratio of lengths $d:h = 11:2$ ✓

(i) $u = \frac{1}{x}$ $\quad du = -\frac{1}{x^2}dx$ $\quad$ Lower limit $x = 1$ $\quad u = \frac{1}{1} = 1$ ✓ (Be generous here even if limits not properly handled)

$x^3 = \frac{1}{u^3}$ $\quad du = -u^2 dx$ $\quad$ Upper limit $\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$

$-\frac{1}{u^2}du = dx$

$$I = \int_{u=1}^{0} \frac{-\frac{du}{u^2}}{1+\frac{1}{u^3}} \quad \begin{matrix}(\times u^3)\\(\times u^3)\end{matrix}$$

✓ SHOW

$$= -\int_1^0 \frac{u\,du}{u^3+1}$$

$$= \int_0^1 \frac{u\,du}{1+u^3} \quad \text{(Reversing limits)}$$

(ii)
$$\int_0^\infty \frac{1}{1+x^3}dx = \int_0^1 \frac{1}{1+x^3}dx + \int_1^\infty \frac{1}{1+x^3}dx \quad \text{(Splitting interval)}$$

$$= \int_0^1 \frac{1}{1+x^3}dx + \int_0^1 \frac{x}{1+x^3}dx$$ ✓ (Using ii and replacing dummy variable u with x)

$$= \int_0^1 \frac{1+x}{1+x^3}dx$$

$$= \int_0^1 \frac{\cancel{1+x}\;1}{\cancel{(1+x)}(1-x+x^2)}dx$$

$$= \int_0^1 \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}dx$$ ✓ $\quad a = \frac{\sqrt{3}}{2}$

$$= \left[\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^1$$

$$= \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}}\left[\frac{\pi}{6} - \left(\frac{-\pi}{6}\right)\right]$$

$$= \frac{2\pi}{3\sqrt{3}}$$

$$= \frac{2\sqrt{3}\pi}{9}$$ ✓

If boys fail to recognise (ii) is NOT evaluating I but has a lower limit of zero.

$I = \frac{\sqrt{3}\pi}{9} - \frac{1}{3}\ln 2$ $\quad$ Harder to evaluate requiring partial fractions 2 marks out of 3.